



MINIMAL MICROWAVE ANISOTROPY FROM PERTURBATIONS INDUCED AT LATE TIMES

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ABSTRACT. Aside from primordial gravitational instability of the cosmological fluid, various mechanisms have been proposed to generate large-scale structure at relatively late times, including, e.g., “late-time” cosmological phase transitions. In these scenarios, it is envisioned that the universe is nearly homogeneous at the time of last scattering and that perturbations grow rapidly sometime after the primordial plasma recombines. On this basis, it was suggested that large inhomogeneities could be generated while leaving relatively little imprint on the cosmic microwave background (MBR) anisotropy. In this paper, we calculate the minimal anisotropies possible in any “late-time” scenario for structure formation, given the level of inhomogeneity observed at present. Since the growth of the inhomogeneity involves time-varying gravitational fields, these scenarios inevitably generate significant MBR anisotropy via the Sachs-Wolfe effect. Moreover, we show that the large-angle MBR anisotropy produced by the rapid post-recombination growth of inhomogeneity is generally *greater* than that produced by the same inhomogeneity grown via gravitational instability. In “realistic” scenarios one can decrease the anisotropy compared to models with primordial adiabatic fluctuations, but only on very small angular scales. The value of any particular measure of the anisotropy can be made small in late-time models, but only by making the time-dependence of the gravitational field sufficiently “pathological”.

I. Introduction

Soon after the discovery of the Microwave Background Radiation (MBR), it was noted that measurements of differences in the MBR temperature in different directions (anisotropy) would provide a sensitive probe of large-scale density inhomogeneities in the universe (Sachs and Wolfe 1967, Rees and Sciama 1968). Recently, the COBE satellite has discovered MBR anisotropy on large angular scales ($> 10^\circ$) (Smoot, *et al.* 1992), and experiments on smaller angular scales have seen signals which may also turn out to be anisotropy in the MBR (Gaier, *et al.* 1992, Devlin, *et al.* 1992, Meyer, *et al.* 1991). While it is clear that large scale density perturbations in the universe will induce MBR anisotropies, the relationship between the anisotropies and the perturbations depends on how the inhomogeneities are produced. The usual assumption is that the density perturbations are primordial, i.e., produced long before recombination, and evolve gravitationally in a relatively simple cosmological fluid composed of, e.g., photons, neutrinos, baryons, and dark matter. If the cosmological matter does have such a simple equation of state, then the inhomogeneities must be primordial, since they will not arise spontaneously in such a fluid. The relation between density inhomogeneities and temperature anisotropies for such primordial perturbations is fairly well understood (Sachs and Wolfe 1967; for a recent introduction, see Peebles 1993), and the implications of the COBE detection for primordial perturbations have been extensively studied (e.g., Wright *et al.* 1992, Efstathiou, Bond, and White 1992). In the simplest case, a spatially flat, Einstein-de Sitter universe with $\Omega = 1$, the



MBR anisotropy gives an imprint of conditions at recombination, when the MBR last scattered; we will call these “primordial” anisotropies.

The other possibility to consider is that the dynamics of some component of the matter in the universe is much more complex than that of a simple fluid, and is able to induce perturbations even at very late times. One class of such models is that of topological defects, such as cosmic strings, textures, or global monopoles. The dynamics of the defects is nontrivial, and as they move around, they induce perturbations in the other matter components (baryons, photons, etc.) via their gravitational attraction. In these models, perturbations are produced both before and after recombination. The induced MBR temperature fluctuations are a mixture of the classical “primordial” anisotropies and anisotropies produced after recombination (Stebbins 1988, Turok and Spergel 1990, Bouchet, Bennett, and Stebbins 1988, Bennett, Stebbins, and Bouchet 1992, Bennett and Rhie 1992).

Another class of models are ones in which perturbations are generated primarily after recombination, or more specifically, in which the inhomogeneity in the gravitational potential increases significantly after recombination. These scenarios include “late-time” phase transitions (Wasserman 1986, Hill, Schramm, and Fry 1989, Hill, Schramm, and Widrow 1991, Press, Ryden, and Spergel 1990, Fuller and Schramm 1992, Frieman, Hill, and Watkins 1992) involving non-trivial scalar field dynamics. In this scenario, one could start with an essentially homogeneous universe at the epoch of last scattering and thus avoid all “primordial” anisotropy. Consequently, it was thought, such models could generate the observed large-scale structure with very small imprint on the MBR, and late phase transitions were posed as alternatives to the standard primordial gravitational instability scenario, which, even after COBE, appears to be on the edge of producing an excessive small-scale MBR anisotropy. However, one does not completely avoid MBR anisotropies in the late-time scenario: in this case, both density perturbations *and* MBR anisotropies are produced after recombination. If the universe is homogeneous at last scattering, then gravitational field perturbations must subsequently grow from zero to their present value in order to account for the observed structure. This time-varying gravitational field will induce MBR anisotropies which may not be very small compared with the “primordial” anisotropies produced in the primordial instability scenario. In other words, MBR anisotropies are an inevitable consequence of the existence of density perturbations today.

In this paper, we calculate the minimal MBR anisotropies associated with scenarios for late structure formation such as a late-time phase transition. More specifically, we determine the minimal anisotropies implied by the boundary conditions of zero inhomogeneity at recombination and a fixed present amplitude of density inhomogeneity inferred from redshift surveys. We do not deal with any particular model of such a phase transition, so we give no predictions for the anisotropies that might be expected in a completely ‘realistic’ scenario. Rather we set a firm lower limit on how small these anisotropies can be, given the observed level of large-scale structure. As we shall see, this lower limit is *not* particularly small when compared to what is expected in scenarios with primordial density inhomogeneities. Moreover, when reasonable smoothness conditions are placed on the evolution of the gravitational potential, the large-angle anisotropy in the late-time scenario is generally *larger* than that for primordial adiabatic fluctuations with the same present level of clustering.

Several estimates of the induced anisotropy have previously been calculated for specific late-time scenarios (Zel’dovich, Kobzarev, and Okun 1974, Stebbins and Turner 1989, Turner, Watkins, and Widrow 1991). These estimates suggest that, in models involving domain walls, the anisotropy may be significantly larger than in the primordial gravitational instability picture. However, methods to fix up the domain wall problem have been suggested (Massarotti 1991, Massarotti and Quashnock 1992), and late-time transitions without domain walls have also been investigated (Press, Ryden, and Spergel 1990, Frieman, Hill, and Watkins 1992). This flux of theoretical developments has motivated the model-independent approach we adopt here.

In §II, we set up the problem of minimizing the temperature correlation function $C(\alpha)$. We analytically study the minimization of the rms variance in the temperature fluctuation, $C(0)$, first assuming the

time-dependence of the gravitational potential is continuous and bounded, and then allowing it to become unbounded. In the latter case, the anisotropy is reduced. This illustrates our general result: if the gravitational potential is constrained to be a smoothly varying function of time, the anisotropy is substantially larger than if the potential is unconstrained. In particular, the large-angle anisotropy for a sufficiently smoothly varying potential is generally larger than for primordial adiabatic perturbations, while the anisotropy for a pathological potential function can be smaller. We also point out the differences expected in the angular dependence of the anisotropy between the primordial and late-time scenarios. In §III, we discuss minimization of the anisotropy for experiments with finite beam-width, and solve the problem numerically for several beam configurations. In §IV, we use analytic fits to the power spectrum of large-scale clustering suggested by recent redshift surveys to normalize the results and make estimates of the minimal anisotropy for different experimental beamwidths. We conclude in §V. The details of some of the numerical computations are relegated to the Appendices.

(A note on nomenclature: subtracting off the monopole and dipole anisotropy, MBR anisotropies on large angular scales can generally be decomposed into two components, one of which can loosely be thought of as arising from the gravitational potential at the surface of last scattering, and the other as due to the time-dependent gravitational potential along the path of the MBR photons since recombination. We are calling these two terms “primordial” and “post-recombination” respectively. In the literature, these are sometimes called the “Sachs-Wolfe” and “Rees-Sciama” effects, but they are both contained in the Sachs-Wolfe (1967) formalism. Nevertheless, we will sometimes partially lapse into this usage as well, and use the terms “primordial” and “usual Sachs-Wolfe” interchangeably. The post-recombination effects we are talking about are rather different than the “Rees-Sciama effect”: Rees and Sciama (1968) studied the anisotropies produced by spherical growing mode density perturbations after recombination, and found differences from the classical Sachs-Wolfe effect only due to the non-linearities in the gravitational instability of the matter. The anisotropies we consider are linear in the amplitude of the present density inhomogeneities.)

II. Minimal Models: Formalism and Analytic Results

As shown by Sachs and Wolfe (1967), a non-uniform gravitational field causes anisotropies in the MBR by making differential changes in the energy of photons. In a spatially flat ($k = 0$) Friedmann-Robertson-Walker (FRW) cosmology, which for simplicity we shall assume throughout, the perturbed metric can be written $g_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})$, where $a(\eta)$ is the FRW scale factor, $\eta_{\mu\nu} = \text{diag}[-1, 1, 1, 1]$ is the Minkowski metric, $h_{\mu\nu}$ is the metric perturbation, and $\eta = x^0$ is the conformal time, $\eta = \int dt/a$. In this geometry, the fractional change in energy of photons moving along the null geodesic $x^\mu(\lambda)$ is

$$\frac{\Delta T}{T} = -\frac{1}{2} \int_{\eta_i}^{\eta_f} n^\mu n^\nu (h_{\mu\nu,0} - 2h_{0\mu,\nu}) d\eta \quad (2.1)$$

to first order in the perturbation. In this expression, $n^\mu = dx^\mu/d\eta$ is the tangent vector of the unperturbed photon trajectory, with components $n^0 = 1$, n^i , where \hat{n} is a spatial unit vector, $\eta_{ij}n^i n^j = 1$, $i, j = 1, 2, 3$. The derivatives $h_{\mu\nu,\alpha}$ are evaluated at points x^μ along the unperturbed photon path.

We can decompose a general metric perturbation $h_{\mu\nu}(x^\mu)$ into scalar, vector, and tensor modes, which evolve independently in linear perturbation theory. These modes are orthogonal in the sense that the expected value of any quadratic measure of the anisotropy is just a sum of the scalar, vector, and tensor components, with no cross terms. Here we only consider quadratic measures of the anisotropy. At present, since there is no compelling evidence for the existence of vector and tensor metric perturbations in the universe, we can estimate the minimal $\Delta T/T$ consistent with observations by considering scalar perturbations alone. Nonzero vector and tensor modes can only increase the anisotropy.

For scalar perturbations, in longitudinal (or conformal Newtonian) gauge, the metric perturbation takes the form $h_{00} = 2\Phi$, $h_{ij} = -2\Psi\eta_{ij}$, where $\Phi(x, \eta)$ and $\Psi(x, \eta)$ are gauge-invariant variables. (In the notation of Bardeen (1980), $\Phi = \Phi_A$ and $\Psi = -\Phi_H$.) Substituting into the Sachs-Wolfe integral (2.1),

setting the metric and temperature fluctuations to zero at recombination, and ignoring the boundary term at the observer, which only contributes to the monopole and dipole anisotropy, we have

$$\frac{\Delta T}{T} = \int_{\eta_r}^{\eta_0} d\eta (\dot{\Psi} + \dot{\Phi}) \quad (2.2)$$

where an overdot denotes differentiation with respect to conformal time. This expression is manifestly gauge-invariant. For a spatially flat ($\Omega = 1$) matter-dominated universe, recombination occurs at conformal time $\eta_r = \eta_0/(1+z_r)^{1/2}$ with $z_r \approx 1100$, and the conformal time today is $\eta_0 = 2H_0^{-1} = 6000h^{-1}$ Mpc. (We use units in which the speed of light $c = 1$.)

If we assume that the universe is dominated by nonrelativistic matter today (or, more generally, if the anisotropic stress of the currently dominating matter vanishes, $\delta T_j^i \sim \delta_j^i$), then the present boundary condition for the two ‘gravitational potentials’ Φ and Ψ is $\Phi_0 = \Psi_0$. As a result, since the two potentials enter the Sachs-Wolfe integral (2.2) identically, for the solution which minimizes $\Delta T/T$ they will be equal for all time, $\Phi = \Psi$. (Note that if the universe is currently dominated by matter with anisotropic stress, one could violate this assumption. In fact, since only Φ contributes to the motion of non-relativistic matter, e.g., galaxies, in this case one could conceivably set $\Psi = 0$ for all time, reducing the anisotropies we will calculate by a factor of two. However, this would presumably require a rather bizarre stress tensor for the dark matter, so we will not consider this possibility further.) Putting in the explicit argument of the potential, we thus have

$$\frac{\Delta T}{T} = 2 \int_{\eta_r}^{\eta_0} d\eta \dot{\Phi}(\mathbf{x}_o - \hat{\mathbf{n}}(\eta_0 - \eta), \eta), \quad (2.3)$$

where \mathbf{x}_o is the observer position coordinate. In the presence of primordial adiabatic perturbations at the surface of last scattering (i.e., what is usually called the Sachs-Wolfe effect), there would also be a term $(\Delta T/T)_{\text{SW}} = (1/3)\dot{\Phi}(\eta_r)$. In the case we are considering here, however, there is no metric or radiation perturbation initially, and the boundary term at emission ($\eta = \eta_r$) is zero.

For the moment we do not subtract off the contribution to the monopole and dipole contribution from (2.3). For experiments that only probe wavenumbers k such that $k\eta_0 \gg 1$, i.e, wavelengths much smaller than the present Hubble radius, this subtraction would not make much difference anyway. Below, we will consider the rms temperature fluctuation for a realistic experimental beam configuration; here, we consider the full temperature autocorrelation function

$$C_{\text{LT}}(\alpha) = \left\langle \frac{\Delta T}{T}(\hat{\mathbf{n}}) \frac{\Delta T}{T}(\hat{\mathbf{m}}) \right\rangle_{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha} = 4 \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\eta' \langle \dot{\Phi}(\mathbf{x}_o - \hat{\mathbf{n}}(\eta_0 - \eta), \eta) \dot{\Phi}(\mathbf{x}_o - \hat{\mathbf{m}}(\eta_0 - \eta'), \eta') \rangle, \quad (2.4)$$

where $\langle \dots \rangle$ denotes an average over all positions \mathbf{x}_o and all directions $\hat{\mathbf{n}}, \hat{\mathbf{m}}$ separated by an angle α .

Taking the Fourier transform of the potential,

$$\Phi(\mathbf{x}, \eta) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \tilde{\Phi}(\mathbf{k}, \eta) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (2.5)$$

results in

$$C_{\text{LT}}(\alpha) = 4 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\eta' \langle e^{i\mathbf{x}_o \cdot (\mathbf{k} - \mathbf{k}')} \rangle_{\mathbf{x}_o} \times \langle e^{-i\mathbf{k} \cdot \hat{\mathbf{n}}(\eta_0 - \eta) + i\mathbf{k}' \cdot \hat{\mathbf{m}}(\eta_0 - \eta')} \rangle_{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha} \langle \dot{\tilde{\Phi}}(\mathbf{k}, \eta) \dot{\tilde{\Phi}}^*(\mathbf{k}', \eta') \rangle. \quad (2.6)$$

Performing the appropriate averages:

$$\langle e^{i\mathbf{x}_o \cdot (\mathbf{k} - \mathbf{k}')} \rangle_{\mathbf{x}_o} = \frac{1}{V} \int d^3 \mathbf{x} e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{x}_o} = \frac{(2\pi)^3}{V} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (2.7)$$

and (see Appendix A)

$$\langle e^{-i\mathbf{k} \cdot \hat{\mathbf{n}}(\eta_0 - \eta) + i\mathbf{k}' \cdot \hat{\mathbf{m}}(\eta_0 - \eta')} \rangle_{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha} = j_0(k \sqrt{(\eta_0 - \eta)^2 + (\eta_0 - \eta')^2 - 2(\eta_0 - \eta)(\eta_0 - \eta') \cos \alpha}) \quad (2.8)$$

gives

$$C_{LT}(\alpha) = \frac{4}{V} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{\eta_r}^{\eta_o} d\eta \int_{\eta_r}^{\eta_o} d\eta' j_0(k \sqrt{(\eta_o - \eta)^2 + (\eta_o - \eta')^2 - 2(\eta_o - \eta)(\eta_o - \eta') \cos \alpha}) \times \langle \tilde{\Phi}(\mathbf{k}, \eta) \tilde{\Phi}^*(\mathbf{k}, \eta') \rangle, \quad (2.9)$$

where $j_0(x) = \sin x/x$ is a spherical Bessel function.

We can factor out the temporal dependence of $\tilde{\Phi}$ for each \mathbf{k} -mode,

$$\tilde{\Phi}(\mathbf{k}, \eta) = \tilde{\Phi}_o(\mathbf{k}) f_{\mathbf{k}}(\eta) \quad \text{with} \quad f_{\mathbf{k}}(\eta_r) = 0, \quad f_{\mathbf{k}}(\eta_o) = 1. \quad (2.10)$$

We will assume that the present day ($\eta = \eta_o$) gravitational potential field is statistically homogeneous and isotropic and hence the expectation of the product, $\tilde{\Phi}_o(\mathbf{k}) \tilde{\Phi}_o^*(\mathbf{k}')$, is given by the gravitational potential power spectrum today, which we call Q ; since Q can only depend on $k = |\mathbf{k}|$, we have

$$\langle \tilde{\Phi}_o(\mathbf{k}) \tilde{\Phi}_o^*(\mathbf{k}') \rangle = (2\pi)^3 Q(k) \delta^{(3)}(\mathbf{k} - \mathbf{k}') \approx V Q(k) \delta_{\mathbf{k}\mathbf{k}'} . \quad (2.11)$$

Thus, the correlation function may be written,

$$C_{LT}(\alpha) = 4 \int \frac{d^3\mathbf{k}}{(2\pi)^3} Q(k) \int_{\eta_r}^{\eta_o} d\eta \int_{\eta_r}^{\eta_o} d\eta' j_0(k \sqrt{(\eta_o - \eta)^2 + (\eta_o - \eta')^2 - 2(\eta_o - \eta)(\eta_o - \eta') \cos \alpha}) \times f_{\mathbf{k}}(\eta) f_{\mathbf{k}}(\eta') . \quad (2.12)$$

Since the mean square anisotropy is just the sum of the anisotropies from the different \mathbf{k} modes, we may optimize the different \mathbf{k} modes (to give minimal $C(\alpha)$) independently. Also note that the optimization just depends on $k = |\mathbf{k}|$ and not on the direction of \mathbf{k} , so the solution for the $f_{\mathbf{k}}$ with the same k will be exactly the same. Hence we can assume that $f_{\mathbf{k}}$ only depends on k and we henceforth use the notation f_k .

By contrast, for the usual Sachs-Wolfe effect (*i.e.*, with primordial adiabatic fluctuations, and purely gravitational evolution thereafter), the temperature correlation function is instead

$$C_{SW}(\alpha) = \frac{1}{9} \int \frac{d^3\mathbf{k}}{(2\pi)^3} Q(k) j_0(2k(\eta_o - \eta_r) \sin(\alpha/2)) . \quad (2.13)$$

To reiterate: for primordial, linear adiabatic fluctuations evolving purely gravitationally in an $\Omega = 1$, matter-dominated universe, $\dot{f}_{\mathbf{k}}(\eta) = 0$ and (2.12) vanishes; in this case, (2.13) gives the entire anisotropy on large angular scales. Here, we are considering the ‘opposite’ case in which the primordial anisotropy $C_{SW}(\alpha) = 0$, and we are seeking to minimize the anisotropy (2.12) arising from the time-dependent gravitational potential, *independent* of any assumptions about gravitational evolution.

To make contact with observations of large-scale structure (see §IV), it is useful to relate $Q(k)$ to $P(k)$, the power spectrum of density fluctuations. Defining the Fourier transform $\delta_{\mathbf{k}}$ of the density field $\delta\rho(\mathbf{x}, \eta)/\bar{\rho}$ as in (2.5), the density power spectrum is defined by analogy with (2.11),

$$\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'}^* \rangle = (2\pi)^3 P(k) \delta^3(\mathbf{k} - \mathbf{k}') . \quad (2.14)$$

By Poisson’s equation, $\nabla^2 \Phi = 4\pi G \rho$, in an $\Omega = 1$ universe,

$$Q(k) = \frac{9}{4} \frac{H_0^4}{k^4} P(k), \quad \int \frac{d^3\mathbf{k}}{(2\pi)^3} Q(k) = \frac{9H_0^4}{8\pi^2} \int dk k^{-2} P(k). \quad (2.15)$$

Minimizing $C(0)$

We first consider the minimization of $C(0) = \langle (\Delta T/T)^2 \rangle$, the rms variance in the temperature fluctuation on the sky; although this is an unmeasurable quantity, it generally sets the scale for the temperature perturbations for a model. In the late-time scenario, from (2.12) and (2.15),

$$\begin{aligned} C_{\text{LT}}(0) &= 4 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} Q(k) \int_{\eta_r}^{\eta_o} d\eta \int_{\eta_r}^{\eta_o} d\eta' j_0(k(\eta' - \eta)) \dot{f}_k(\eta) \dot{f}_k(\eta') \\ &= \frac{9H_0^4}{2\pi^2} \int dk k^{-2} P(k) \int_{\eta_r}^{\eta_o} d\eta \int_{\eta_r}^{\eta_o} d\eta' j_0(k(\eta' - \eta)) \dot{f}_k(\eta) \dot{f}_k(\eta') \\ &\equiv \frac{9H_0^4}{2\pi^2} \int dk k^{-2} P(k) I_k[f_k] , \end{aligned} \quad (2.16)$$

while, for primordial adiabatic perturbations, the Sachs-Wolfe result (2.13) is

$$C_{\text{SW}}(0) = \frac{1}{9} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} Q(k) = \frac{H_0^4}{8\pi^2} \int dk k^{-2} P(k). \quad (2.17)$$

In general, for power spectra that behave as $P(k) \propto k^n$ for small k , the integrals (2.16) and (2.17) will diverge at long wavelengths if $n < 2$. Since $C(0)$ is not an observable, this divergence is not problematic. For example, we would obtain finite results if we calculate physical quantities such as $\sqrt{2(C(0) - C(\alpha))}$, the rms temperature difference measured by a two-beam experiment with a beam-throw of angle α . (In addition, we should take into account the finite width of any real beam.) Alternately, we could subtract off the unmeasurable monopole and dipole terms from $C(\alpha)$ (see below). Here, we are interested in comparing the usual Sachs-Wolfe and the late-time anisotropies in k -space, so we need not perform the divergent wavenumber integrals.

If we wish to minimize the functional (2.16), it remains to find a set of optimal solutions $f_k(\eta)$, given by the minimization of

$$I_k[f_k] = \int_{\eta_r}^{\eta_o} d\eta \int_{\eta_r}^{\eta_o} d\eta' g_k(\eta - \eta') \dot{f}_k(\eta) \dot{f}_k(\eta'), \quad \text{where} \quad g_k(\eta - \eta') \equiv \frac{\sin k(\eta' - \eta)}{k(\eta' - \eta)}, \quad g_k(x) = g_k(-x), \quad (2.18)$$

with respect to the function f_k , which must satisfy the boundary conditions (2.10).

High and Low Frequency Limits: The Linear Model

To gain some insight into the level of minimal anisotropy expected, we can analytically explore the integral I_k in the short and long wavelength limits, which correspond roughly to $k\eta_o$ much larger and smaller than one respectively. (It is useful to recall the conversion $k = (6.67 \times 10^{-5} k\eta_o) \text{ h Mpc}^{-1}$.) If the function f is well approximated by its Taylor series close to every point, then in the short wavelength limit $k \rightarrow \infty$, we may make the substitution in the integral (2.18)

$$g_k(\eta - \eta') \equiv \frac{\sin k(\eta' - \eta)}{k(\eta' - \eta)} \rightarrow \pi \delta(k(\eta - \eta')). \quad (2.19)$$

In this limit, I_k reduces to the action functional for a free particle with ‘coordinate’ f_k , and the condition for an extremal time history is

$$\ddot{f}_k = 0, \quad \text{i.e.,} \quad f_k(\eta) = f_k^{\text{lin}} \equiv \frac{\eta - \eta_r}{\eta_o - \eta_r} \approx \frac{\eta}{\eta_o}. \quad (2.20)$$

In this case, the minimal temperature correlation function becomes

$$C_{LT}(0) \rightarrow 4\pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{Q(k)}{k\eta_d} \quad \text{as } k \rightarrow \infty, \quad (2.21)$$

where $\eta_d = \eta_o - \eta_r \approx \eta_o$. Comparing with the usual Sachs-Wolfe expression (2.17), we see that, for the same amplitude of present structure $P(k)$, the minimal late-time anisotropy is only smaller than the primordial anisotropy in the wavenumber range

$$k\eta_d > 36\pi \quad \text{or} \quad \lambda < \frac{\eta_d}{18}. \quad (2.22a)$$

In the spatially flat cosmology we have assumed, this corresponds to

$$k^{-1} < 53 h^{-1} \text{Mpc} \quad \text{or} \quad \lambda < 333 h^{-1} \text{Mpc}. \quad (2.22b)$$

corresponding to a few degrees on the sky. (This lengthscale is comparable to the largest scales currently probed by redshift surveys, Cf. Fig. 3 below.) Thus, for the very large-scale perturbations which are now starting to be probed by COBE and other experiments, one cannot really do better than primordial adiabatic perturbations in minimizing the anisotropy. We can understand results (2.21-22) heuristically, by considering the contribution from perturbations of comoving wavelength λ to the rms anisotropy. For primordial adiabatic perturbations, $(\Delta T/T)_{SW} = (1/3)\Phi$, where the potential fluctuation on scale λ is $\Phi_\lambda \sim (\delta\rho/\rho)_\lambda(\lambda/t_0)^2$. For late-time perturbations, on the other hand, the anisotropy is proportional to the integrated time-derivative of the potential, $(\Delta T/T)_{LT} \sim \int \dot{\Phi}_\lambda dt \sim \Phi_{0,\lambda}(\lambda/t_0)N_\lambda^{1/2}$, where $N_\lambda \sim t_0/\lambda$ is roughly the number of lumps of size λ between the observer and the hypersurface when the potential began to increase (assumed to be at $z \gtrsim 1$ here). As a result, we find $(\Delta T/T)_{LT} \sim \Phi_{0,\lambda}(\lambda/t_0)^{1/2}$; note that this wavelength dependence agrees with (2.21). Comparing this late-time expression with the Sachs-Wolfe anisotropy above, we see that the minimal late-time anisotropy is smaller than the primordial anisotropy only for small wavelengths, $(\lambda/H_0^{-1})^{1/2} \lesssim 1/3$, in agreement with (2.22). At large wavelengths, the minimal late-time anisotropy is *greater* than the standard Sachs-Wolfe result because, for primordial adiabatic perturbations, the anisotropy is only 1/3 of the gravitational potential fluctuation. This famous 1/3 factor arises from a partial cancellation between the gravitational redshift, Φ , and the varying radiation temperature at recombination $(\Delta T/T)_\gamma = -(2/3)\Phi$ (for superhorizon perturbations in Newtonian gauge). In contrast, for the late-time scenario there is no corresponding cancellation since the last-scattering surface is unperturbed.

The linear form (2.20), $f_k^{\text{lin}}(\eta) = (\eta - \eta_r)/\eta_d$, is a good paradigm for a slowly increasing potential fluctuation, and we shall use it as a fiducial reference against which to compare other results. Let us see what it gives for arbitrary k . In this case we have

$$\begin{aligned} I_k^{\text{lin}} &= \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x-y))}{k\eta_d(x-y)} \\ &= 2 \left[\frac{k\eta_d \text{Si}(k\eta_d) - (1 - \cos(k\eta_d))}{(k\eta_d)^2} \right] \end{aligned} \quad (2.23)$$

where Si is the sine integral function (see Appendix A). Since $\text{Si}(\infty) = \frac{\pi}{2}$ and $\text{Si}(x) = x + \mathcal{O}(x^3)$ for small x , we find the limits

$$I_k^{\text{lin}} \rightarrow \frac{\pi}{k\eta_d} \quad \text{for } k\eta_d \rightarrow \infty \quad \text{and} \quad I_k^{\text{lin}} \rightarrow 1 \quad \text{for } k\eta_d \rightarrow 0. \quad (2.24)$$

The $k\eta_d \rightarrow \infty$ limit here agrees with eqn.(2.21).

Angular Structure for the Linear Model

Although the absolute values of the temperature fluctuations in late-time models may be comparable to those in standard primordial adiabatic perturbation scenarios, the angular dependence of the anisotropy for the two cases can be quite different. To see this, we compare the quantities $J(k, \alpha)$ that are integrated with the power spectrum in the temperature correlation function:

$$\begin{aligned} C(0) - C(\alpha) &= \frac{9H_0^4}{8\pi^2} \int dk k^{-2} P(k) J(k, \alpha) \\ J_{\text{SW}}(k, \alpha) &= \frac{1}{9} [1 - j_0(2k\eta_0 \sin(\alpha/2))] \\ J_{\text{LT}}(k, \alpha) &= 4 \int_{\eta_r}^{\eta} d\eta \int_{\eta_r}^{\eta'} d\eta' \left[j_0(k(\eta - \eta')) - j_0(k\sqrt{(\eta_0 - \eta)^2 + (\eta_0 - \eta')^2 - 2(\eta_0 - \eta)(\eta_0 - \eta') \cos \alpha}) \right] \end{aligned} \quad (2.25)$$

where we have used the linear model $f_k^{\text{lin}}(\eta)$ in J_{LT} . In Fig. 1, we plot J_{LT} and J_{SW} as functions of k for different values of the angle α . Two differences between them are immediately apparent. First, consider the behavior at large k : in this regime, the contribution to J is dominated by the $\alpha = 0$ part, $J(k, \alpha) \rightarrow 4I_k$. For the Sachs-Wolfe case, with $I_k = 1/36$, the contribution does not fall off at these small scales. For the late-time model, $J(k, \alpha) \approx 4I_k \rightarrow 4\pi/k\eta_0$ for large k (for a particular value of the angle α , this limit is appropriate for all k beyond the maximum of J_{LT} for that angle—i.e., beyond the scale that contributes the greatest to anisotropies of that angular separation.) Because of this difference, power spectra with significant small-scale power may imply greater anisotropies for primordial perturbations than in the late-time case.

Of greater import, however, are the values of J on intermediate scales. Although the maxima of J are located at similar values of k for both cases, the maximal values of J are much greater for the late-time scenario. On very small angles, the contribution of intermediate scales to the primordial Sachs-Wolfe effect is larger than for the late-time scenario; however, as the angle increases, the contribution to the anisotropy for the late-time model continues to grow, whereas in the Sachs-Wolfe case the maximum value of J remains approximately constant, independent of α . For a given power spectrum, then, the anisotropy at a given angular scale will generically be larger for a late-time scenario than for primordial perturbations, unless the evolution of the potential is specifically chosen to decrease the anisotropy on that scale.

Low Frequencies

In the other extreme of low frequencies, or long wavelengths, from (2.18) we have $g_k \rightarrow 1$, so the dependence of I_k on the functions f_k drops out; from the boundary conditions (2.10), we then find $I_k \rightarrow 1$, which agrees with and generalizes the low frequency result for I_k^{lin} in (2.24). In this limit, from (2.16) we have

$$C_{\text{LT}}(0) \rightarrow 4 \int \frac{d^3\mathbf{k}}{(2\pi)^3} Q(k) = \frac{9H_0^4}{2\pi^2} \int dk k^{-2} P(k) \quad \text{as } k \rightarrow 0, \quad (2.26)$$

which is 36 times the value for primordial adiabatic perturbations, $C_{\text{SW}}(0)$, in eqn. (2.17). That is, in the long-wavelength limit, we expect the rms anisotropy to be 6 times larger for the late-time scenario than for primordial adiabatic fluctuations; it is worth noting that, for primordial isocurvature fluctuations, the rms Sachs-Wolfe anisotropy is also six times larger than for adiabatic perturbations, $(\Delta T/T)_{\text{SW, isoc}} = 2\Phi$.

Based on this discussion, one may be tempted to conclude that the behavior $\min[I_k] \rightarrow 1$ as $k \rightarrow 0$ is generic. However this is not the case. To see this, consider minimizing I_k for the small space of functions

$$f_k(\eta) = \frac{\eta - \eta_r}{\eta_d} + a_k \sin\left(2\pi \frac{\eta - \eta_r}{\eta_d}\right). \quad (2.27)$$

In this case, I_k is given by

$$I_k = I_{\text{lin},k} + a_k I_1 + a_k^2 I_2, \quad (2.28)$$

where $I_{\text{lin},k}$ is given by (2.23), and (see Appendix A)

$$\begin{aligned} I_1 &= 2\pi \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x-y))}{k\eta_d(x-y)} (\cos(2\pi x) + \cos(2\pi y)) \\ &= \frac{2}{k\eta_d} \left(\text{Ci}(|k\eta_d + 2\pi|) - \text{Ci}(|k\eta_d - 2\pi|) - \ln \left| \frac{k\eta_d + 2\pi}{k\eta_d - 2\pi} \right| \right), \\ &\longrightarrow -\frac{1}{3\pi} (k\eta_d)^2 + \mathcal{O}((k\eta_d)^4) \quad \text{as } k \rightarrow 0 \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} I_2 &= (2\pi)^2 \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x-y))}{k\eta_d(x-y)} \cos(2\pi x) \cos(2\pi y) \\ &= \frac{\pi}{k\eta_d} \left(\text{Ci}(|k\eta_d + 2\pi|) - \text{Ci}(|k\eta_d - 2\pi|) - \ln \left| \frac{k\eta_d + 2\pi}{k\eta_d - 2\pi} \right| \right) \\ &\quad + \frac{2\pi^2}{k\eta_d} (\text{Si}(k\eta_d + 2\pi) + \text{Si}(k\eta_d - 2\pi)) + \frac{(2\pi)^2}{(2\pi)^2 - (k\eta_d)^2} (1 - \cos(k\eta_d)) \\ &\longrightarrow \frac{1}{20\pi^2} (k\eta_d)^4 + \mathcal{O}((k\eta_d)^6) \quad \text{as } k \rightarrow 0 \end{aligned} \quad (2.30)$$

Minimizing I_k with respect to a_k gives

$$a_k = -\frac{I_1}{2I_2} \rightarrow \frac{10\pi}{3(k\eta_d)^2} + \mathcal{O}((k\eta_d)^0) \quad \text{as } k \rightarrow 0, \quad (2.31)$$

and hence

$$\min_a [I_k] = I_{\text{lin},k} - \frac{I_1^2}{4I_2} \rightarrow \frac{4}{9} + \mathcal{O}((k\eta_d)^2) \quad \text{as } k \rightarrow 0. \quad (2.32)$$

The small- k limit of the minimizing integral I_k is $4/9$ of the value estimated above, so the corresponding minimal $C(0)$ is $4/9$ of the value given in (2.26). Thus, for the class of functions (2.27), in the long wavelength limit the minimal late-time rms anisotropy is 4 rather than 6 times larger than for primordial adiabatic fluctuations. However, in order to achieve this limit, the coefficient a_k diverges, and therefore the function f_k becomes unbounded, as $k \rightarrow 0$. This pathological behavior is not what one would expect for the gravitational potential evolution in a ‘realistic’ late-time scenario, but it provides a lower bound on the anisotropy in the long wavelength limit. One can generalize this procedure by adding m terms of the form $a_{k,m} \sin(\pi m(\eta - \eta_t)/\eta_d)$ to f_k in eqn. (2.27); we shall use this technique below in our numerical work. Extending the sum to larger m further reduces the small- k limit of I_k from the value we found for $m = 2$. Also note that in the short wavelength limit, $k \rightarrow \infty$, $a_{k,2} \rightarrow 0$ and we retrieve the linear solution (2.20). However, as m is increased, the $a_{k,m}$ fall off more slowly with increasing k .

The lesson we draw from this example is that the minimal late-time anisotropy can be substantially smaller than that in the linear model f_k^{lin} which we have been focusing on, but that this reduction is achieved at the cost of introducing a potential function $f_k(\eta)$ which varies rather wildly with conformal time.

Multipole Expansion

It is often convenient to expand the temperature correlation function in angular multipoles,

$$C(\alpha) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_{\ell}(\cos \alpha) C_{\ell} \quad \text{where} \quad C_{\ell} = 2\pi \int_{-1}^1 d\cos \theta P_{\ell}(\cos \theta) C(\theta), \quad (2.33)$$

and the P_{ℓ} ’s are Legendre polynomials. For the late-time scenario, the angular integral in (2.33) decouples the two integrals over conformal time in Eq.(2.12), and the resulting angular power spectrum is

$$C_{\ell, \text{LT}} = \frac{18H_0^4}{\pi} \int dk k^{-2} P(k) \left[\int_{\eta_*}^{\eta_0} d\eta j_{\ell}(k(\eta_0 - \eta)) \dot{f}_k(\eta) \right]^2, \quad (2.34)$$

where $j_\ell(x)$ is the spherical Bessel function. Again, we can compare to the Sachs-Wolfe anisotropy for primordial, adiabatic perturbations,

$$C_{\ell, \text{SW}} = \frac{H_0^4}{2\pi} \int dk k^{-2} P(k) [j_\ell(k\eta_0)]^2. \quad (2.35)$$

The angular spectrum (2.34) points to two curious features of the minimization of $C_{\text{LT}}(0)$ which was missed above. First, it is apparent that by judicious choice of the functions $f_k(\eta)$ we can make any particular multipole moment C_ℓ vanish. (We will see an example of this in §IV below.) Once a given multipole is set to zero, this specifies all the f_k , and the other multipoles will in general be non-zero. However, consider the unphysical case in which the potential turns on instantaneously at conformal time $\eta = \eta_f$, i.e., $f_k(\eta) = \theta(\eta - \eta_f)$. Then the bracketed expression in (2.34) becomes $j_\ell(k(\eta_0 - \eta_f))$. In particular, in the limit $\eta_f \rightarrow \eta_0$, this expression vanishes for all $\ell > 0$. Thus, if the potential turned on instantaneously at the present time, we have $C_{\text{LT}}(0) = 36C_{\text{SW}}(0)$, in agreement with (2.26), but the anisotropy is hidden in the unobservable monopole term C_0 . (This possibility did not appear in the high and low frequency limit discussion of $C(0)$ above, because we did not subtract off the monopole term there.) While a useful theoretical foil, this example is not of direct physical interest: an instantaneous turn-on of the potential violates causality. More generally, structure on a given scale cannot be made in less than a light-crossing time for that scale. Furthermore, we know that structure existed before the present time: conservatively, the gravitational potential corresponding to the growing mode density fluctuation was essentially in place by a redshift $z_f \gtrsim 3$, corresponding to $\eta_f \lesssim 0.5\eta_0$. This constraint, which we will impose in calculating observables below, implies that the $\ell \neq 0$ multipoles will be non-zero, although the higher multipoles may be relatively suppressed.

To see this, consider the instantaneous turn-on of the potential at z_f . Then the contribution of the k -mode waveband to the ℓ th moment, relative to that for the primordial Sachs-Wolfe anisotropy, is

$$\frac{dC_{\ell, \text{LT}}/d\ln k}{dC_{\ell, \text{SW}}/d\ln k} = 36 \left[\frac{j_\ell(k(\eta_0 - \eta_f))}{j_\ell(k\eta_0)} \right]^2 \rightarrow 36 \left[1 - (1 + z_f)^{-1/2} \right]^{2\ell} \quad \text{as } k \rightarrow 0, \quad (2.36)$$

for $\ell \neq 0$. For example, with $z_f = 3$, in the long wavelength limit, the late-time quadrupole is larger than that for primordial perturbations, but the octopole and higher moments are smaller. On the other hand, if the potential turns on rapidly at $z_f \gg 1$, as would be expected in most plausible late-time models, then $C_{\ell, \text{LT}} = 36C_{\ell, \text{SW}}$, independent of k : in this case, the multipole structure in the late-time and primordial scenarios is identical, but the late-time anisotropy is 6 times larger.

III. Minimization for Finite Beams: Formalism and Numerical Results

In the previous section, we studied the temperature correlation function under the assumption of an infinitesimally small beamwidth. We now wish to consider the expectation of the mean square anisotropy of a realistic beam configuration when averaged over the sky and averaged over all observers. This is given by some rotationally invariant quadratic moment of the temperature field, which, using (2.4), may be written

$$\begin{aligned} \mathcal{T} &= \int \frac{d^2 \hat{\mathbf{m}}}{4\pi} W(\cos \alpha) \left\langle \frac{\Delta T}{T}(\hat{\mathbf{n}}) \frac{\Delta T}{T}(\hat{\mathbf{m}}) \right\rangle_{\mathbf{x}_0, \hat{\mathbf{n}}} = \frac{1}{2} \int_{-1}^1 d \cos \alpha W(\cos \alpha) C(\alpha) \\ &= 4 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} Q(k) \int_{\eta_f}^{\eta_0} d\eta \int_{\eta_f}^{\eta_0} d\eta' \left[\int_{-1}^1 \frac{dy}{2} W(y) j_0(k \sqrt{(\eta' - \eta_0)^2 + (\eta - \eta_0)^2 - 2y(\eta' - \eta_0)(\eta - \eta_0)}) \right] \\ &\quad \times f_k(\eta) f_k(\eta'). \end{aligned} \quad (3.1)$$

Here, as before, α is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$, and $W(\cos \alpha)$ is a weighting function which is determined by the beam configuration. For the temperature correlation function $C(\alpha)$ we have used (2.12), except that

the upper limits on the η and η' integrals are η_t instead of η_o . This replacement incorporates the assumption that the potential perturbations were constant from the epoch η_t until the present, η_o , that is, that $\dot{\Phi}(\eta) = 0$ for $\eta > \eta_t$. This corresponds to the statement that the growing mode density fluctuations were in place at some minimum redshift z_t .

Our task is to find the functions $f_k(\eta)$ which minimize \mathcal{T} . Following eqns. (2.16, 2.18), we use the somewhat more compact notation:

$$\begin{aligned} \mathcal{T} &= 4 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} Q(k) I_k & I_k[f_k] &= \int_{\eta_r}^{\eta_t} d\eta \int_{\eta_r}^{\eta_t} d\eta' g_k(\eta, \eta') \dot{f}_k(\eta) \dot{f}_k(\eta') \\ g_k(\eta, \eta') &= \frac{1}{2} \int_{-1}^1 dy W(y) j_0 \left(k \sqrt{(\eta' - \eta_o)^2 + (\eta - \eta_o)^2 - 2y(\eta' - \eta_o)(\eta - \eta_o)} \right) \end{aligned} \quad (3.2)$$

Note that from the definition of g_k , we have $g_k(\eta, \eta') = g_k(\eta', \eta)$, and in any case the integral I_k only depends on the symmetric part of g_k .

Explicitly, the condition that f_k gives an extremum of the functional I_k is that

$$\left. \frac{d}{d\epsilon} I_k[f_k + \epsilon \Delta] \right|_{\epsilon=0} = 0 \quad (3.3)$$

for all functions $\Delta(\eta)$ which are zero at the endpoints, $\Delta(\eta_r) = \Delta(\eta_t) = 0$. Thus

$$\begin{aligned} \left. \frac{d}{d\epsilon} I_k[f_k + \epsilon \Delta] \right|_{\epsilon=0} &= 2 \int_{\eta_r}^{\eta_t} d\eta \int_{\eta_r}^{\eta_t} d\eta' g_k(\eta, \eta') \dot{f}_k(\eta) \dot{\Delta}(\eta') \\ &= -2 \int_{\eta_r}^{\eta_t} d\eta \int_{\eta_r}^{\eta_t} d\eta' g_{k,2}(\eta, \eta') \dot{f}_k(\eta) \Delta(\eta') + 2 \int_{\eta_r}^{\eta_t} d\eta \dot{f}_k(\eta) [\Delta(\eta_t) g_k(\eta, \eta_t) - \Delta(\eta_r) g_k(\eta, \eta_r)] \\ &= 2 \int_{\eta_r}^{\eta_t} d\eta \int_{\eta_r}^{\eta_t} d\eta' g_{k,2}(\eta' - \eta) \dot{f}_k(\eta) \Delta(\eta') \end{aligned} \quad (3.4)$$

where we use the notation $g_{k,2}$ to mean differentiation with respect to the second argument, $g_{k,2} = \partial g(x, y) / \partial y$, and we have used the symmetry of g_k . Since this is true for all variations Δ , we see that this is equivalent to the condition

$$\int_{\eta_r}^{\eta_t} d\eta g_{k,2}(\eta, \eta') \dot{f}_k(\eta) = g_{k,2}(\eta_t, \eta') - \int_{\eta_r}^{\eta_t} d\eta g_{k,12}(\eta, \eta') f_k(\eta) = 0 \quad \forall \eta' \in (\eta_r, \eta_t). \quad (3.5)$$

This expression is of the form (operator) $\times f_k$ = function. If we can find the inverse of this linear integral operator, then we can solve for f_k^{\min} , which minimizes I_k . Using this equation in the definition (3.2) of I_k , integrating by parts, and using the boundary conditions (2.10), we find that for a true extremal function

$$\min[I_k] = g_k(\eta_t, \eta_t) - \int_{\eta_r}^{\eta_t} d\eta g_{k,1}(\eta, \eta_t) f_k^{\min}(\eta). \quad (3.6)$$

The details of the numerical procedure we use to find f_k^{\min} are given in Appendix B. We replace the continuous conformal time interval $\eta \in (\eta_r, \eta_t)$ by a grid of N points η_i , and use the trapezoidal rule approximation to convert the double integral for I_k into a double sum. Following the discussion of §II, we choose $\eta_t = 0.5\eta_o$ and to excellent approximation set $\eta_r = 0$. We generally find that the set of $f_k^{\min}(\eta_i)$ is discontinuous at the scale of the grid. To impose a smoothness cutoff independent of the grid size, we therefore expand the f_k in sine waves, generalizing (2.27),

$$f_k(\eta) = \frac{\eta}{\eta_t} + \sum_{m=1}^n a_{k,m} \sin \left(m\pi \frac{\eta}{\eta_t} \right), \quad (3.7)$$

with integer m . As required, this satisfies the boundary conditions $f_k(0) = 0$, $f_k(\eta_t) = 1$. By taking $n < N$, we restrict $f_k(\eta)$ from varying significantly over the grid scale. For the results shown below, we used $n = 70$ or $n = 100$ sine waves and $N = 200$ grid points. As a check, we have also expanded the $f_k(\eta)$ as polynomials of order n (with appropriate boundary conditions) and minimized the I_k with respect to the coefficients using an $n - 1$ -dimensional simplex algorithm. The results thus obtained are consistent with the analytic and numerical techniques discussed above (although this “brute force” method generally finds I_k larger than the more rigorous discretization).

Before we can proceed to minimize I_k , we must choose the particular weighting, W , for which we wish to minimize \mathcal{T} . The functions f_k which minimize \mathcal{T} are unlikely to be the same for different functions $W(\cos \alpha)$. First, to make contact with the discussion of §II, consider minimizing the mean square anisotropy at a point, $C(0) = \langle (\Delta T/T)^2 \rangle$. In this case, the appropriate window function is $W(y) = \lim_{\epsilon \rightarrow 0^+} 2\delta(y - 1 + \epsilon)$, where the small positive ϵ guarantees that the full δ -function is included in the integral. Substituting this into (3.2), we find $g_k(\eta, \eta') = \sin(k(\eta - \eta'))/k(\eta - \eta')$, in agreement with (2.18).

To minimize other quantities more directly related to experimental observations, it is convenient to again expand in multipoles. Using the multipole expansion (2.33), we can write

$$\mathcal{T} = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} W_{\ell} C_{\ell} \quad (3.8)$$

where

$$W(x) = \sum_{\ell=0}^{\infty} (2\ell+1) W_{\ell} P_{\ell}(x) \quad \text{and} \quad W_{\ell} = \frac{1}{2} \int_{-1}^1 W(x) P_{\ell}(x) dx. \quad (3.9)$$

In terms of W_{ℓ} we may rewrite the expression for g_k in (3.2):

$$\begin{aligned} g_k(\eta, \eta') &= \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) W_{\ell} \int_{-1}^1 dy P_{\ell}(y) \frac{\sin k \sqrt{(\eta' - \eta_0)^2 + (\eta - \eta_0)^2 - 2y(\eta' - \eta_0)(\eta - \eta_0)}}{k \sqrt{(\eta' - \eta_0)^2 + (\eta - \eta_0)^2 - 2y(\eta' - \eta_0)(\eta - \eta_0)}} \\ &= \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) W_{\ell} \frac{\pi}{k \sqrt{(\eta' - \eta_0)(\eta - \eta_0)}} J_{\ell+\frac{1}{2}}(k(\eta' - \eta_0)) J_{\ell+\frac{1}{2}}(k(\eta - \eta_0)) \\ &= \sum_{\ell=0}^{\infty} (2\ell+1) W_{\ell} j_{\ell}(k(\eta' - \eta_0)) j_{\ell}(k(\eta - \eta_0)) \end{aligned} \quad (3.10)$$

where the J_n are Bessel functions, and $j_{\ell}(x) = \sqrt{\pi/(2x)} J_{\ell+\frac{1}{2}}(x)$ are spherical Bessel functions.

Minimization for a finite beam experiment

Although we discussed the minimization of the rms temperature anisotropy $C(0)$ in §II, we also pointed out that it is not an observable quantity. Instead, one measures the temperature fluctuation over some finite region of the sky determined by the beam pattern of the instrument. In many cases, the instrument beam is roughly of Gaussian form. Given the intrinsic temperature pattern on the sky, one can construct the sky temperature pattern convolved with the beam. For a Gaussian beam of width σ (radians), the two-point correlation function of this beam-smoothed pattern is given by

$$C(\alpha, \sigma) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} e^{-\ell(\ell+1)\sigma^2} C_{\ell} P_{\ell}(\cos \alpha). \quad (3.11)$$

Thus, to minimize $\mathcal{T} = C(\alpha, \sigma)$ we would use

$$W_{\ell} = e^{-\ell(\ell+1)\sigma^2} P_{\ell}(\cos \alpha) \quad (3.12)$$

in (3.10) to determine $g_k(\eta, \eta')$.

As an example which we will use below, consider the recent COBE DMR observations (Smoot, et al. 1992). The DMR beam is approximately Gaussian with a 7° FWHM; since $\sigma = 0.43$ FWHM, this implies $\sigma_{COBE} = 5.2 \times 10^{-2}$. The DMR team published three results of interest: the quadrupole anisotropy; the correlation function (3.11) with terms $\ell = 0, 1, 2$ removed; and the rms fluctuations smoothed on 10° , with the monopole and dipole removed. The latter result is the most useful for us, and we can express it via (3.11) as $C(0, \sigma[10^\circ])$, where the beam-width corresponding to the 10° FWHM is $\sigma[10^\circ] = \sqrt{2}\sigma_{COBE}$, and we remove the $\ell = 0, 1$ terms from (3.11).

In §IV, we will give results for $C(0, \sigma)$ for a variety of beamwidths σ ; from (2.34), (3.11), and (3.12), we have

$$\begin{aligned} C(0, \sigma) &= \frac{9H_0^4}{2\pi^2} \int dk k^{-2} P(k) \sum_{\ell=2} (2\ell+1) e^{-\ell(\ell+1)\sigma^2} \left[\int_0^\pi d\eta j_\ell(k(\eta_0 - \eta)) \dot{f}_k(\eta) \right]^2 \\ &= \frac{9H_0^4}{2\pi^2} \int dk k^{-2} P(k) I_k(0, \sigma) \end{aligned} \quad (3.13)$$

In Fig. 2, we show the minimizing integral $\min[I_k(0, \sigma)]$ for $C(0, \sigma[1^\circ])$ and $C(0, \sigma[10^\circ])$ as a function of $k\eta_0$, for the late-time scenario (minimized according to eqn. 3.7, and shown as the points denoted LT in the figure) and for primordial adiabatic perturbations (curves denoted SW). This shows that, for a given power spectrum $P(k)$, the minimal late-time anisotropy on 10° is smaller than for primordial adiabatic perturbations, unless the spectrum were narrowly peaked around $k\eta_0 \sim 20$, in which case they would be similar. On 1° , the situation from Fig. 2 is less obvious: if $P(k)$ has little power at $k\eta_0 \lesssim 20$, then the minimal late-time anisotropy on this scale could be larger than the primordial anisotropy, but if there is significant power on these large scales, the larger Sachs-Wolfe I_k in this small- k region would lead to larger relative anisotropy for primordial fluctuations. Note that the corresponding potential functions $f_k(\eta)$ for the minimizing late-time scenarios are wildly oscillating functions of η (see Appendix B and Fig. 6). If we instead constrained $f_k(\eta)$ to be a more gently varying function as in §II (e.g., eqn. 2.20), the resulting I_k 's in the late-time models would lie above the Sachs-Wolfe results over the range of k shown in Fig. 2 (Cf. Fig. 1).

Minimization for a Multiple Beam Switching Experiment

A common type of MBR anisotropy experiment is a switching experiment which, in its simplest form, consists of measuring the temperature convolved with a Gaussian beam at 2 or 3 evenly spaced points on the sky. For reference, we give here the corresponding window functions. For a 2-beam experiment with Gaussian beam-width σ , and beam throw α , one should take

$$\mathcal{T} = \frac{1}{2} \left\langle \left(\frac{\Delta T}{T}_1 - \frac{\Delta T}{T}_2 \right)^2 \right\rangle = C(0, \sigma) - C(\alpha, \sigma), \quad (3.14)$$

and hence

$$W_\ell = e^{-\ell(\ell+1)\sigma^2} (1 - P_\ell(\cos \alpha)). \quad (3.15)$$

For a three-beam experiment with throw α between adjacent beams, one can take

$$\mathcal{T} = \frac{1}{6} \left\langle \left(\frac{\Delta T}{T}_1 - 2\frac{\Delta T}{T}_2 + \frac{\Delta T}{T}_3 \right)^2 \right\rangle = C(0, \sigma) - \frac{4}{3}C(\alpha, \sigma) + \frac{1}{3}C(2\alpha, \sigma), \quad (3.16)$$

and hence

$$W_\ell = e^{-\ell(\ell+1)\sigma^2} \left(1 - \frac{4}{3}P_\ell(\cos \alpha) + \frac{1}{3}P_\ell(\cos 2\alpha) \right). \quad (3.17)$$

The number quoted for " $\Delta T/T$ " for these experiments is $\sqrt{\mathcal{T}}$ times some factor, where the factor used may vary from experiment to experiment.

IV. Power Spectra and Results

At present, the COBE DMR results are the only probe of the power spectrum on scales larger than a few hundred $h^{-1}\text{Mpc}$. Under the standard hypothesis of primordial adiabatic perturbations, COBE provides direct information on the large-scale primordial power spectrum through the Sachs-Wolfe effect, e.g., eqn.(2.13), and Smoot, et al. (1992) find $P(k) \propto k^n$, with $n \simeq 1 \pm 0.5$. However, if we discard the assumption of primordial perturbations, the results of a MBR anisotropy experiment can no longer be used to determine $P(k)$ in the absence of a specific model for the evolution of the gravitational potential. Specifically, the consistency of the COBE results with the inflationary prediction of a Harrison-Zel'dovich spectrum, $P(k) \propto k$ on large scales, could be an artifact of some other power spectrum along with suitably chosen k -dependence for the evolution functions $f_k(\eta)$ (Cf. Fig. 1).

Complementing the COBE results, there have recently been several determinations of the galaxy power spectrum from catalogs derived from the IRAS survey and others. However, these observations do not determine the power spectrum on scales large enough to overlap those probed by COBE. In particular, while COBE probes the (primordial) shape of the spectrum on very large scales, galaxy observations only extend up to scales of order $100 h^{-1} \text{Mpc}$, where significant processing of the primordial spectrum has taken place. If we do interpret the COBE results as a Sachs-Wolfe probe of primordial perturbations, then the resulting COBE spectrum (e.g., Harrison-Zel'dovich) need only be matched onto the smaller scale galaxy observations (modulo such crucial factors as biasing and selection effects). In the late-time scenario, however, the shape of the power spectrum on large scales is not uniquely fixed, but only constrained, by the COBE observations.

In the previous section, we compared the contributions to the anisotropies for the late-time and primordial scenarios from a given wavenumber k , for the same amount of power $P(k)$. Here, we integrate these contributions over several phenomenological power spectra to make predictions for the observable anisotropy for different beam configurations. These phenomenological spectra include two models which approach the Harrison-Zel'dovich form at large scales, but which differ on small scales: one is the standard cold dark matter (CDM) spectrum, and the other is an analytic fit to the QDOT galaxy power spectrum. However, since we are not assuming primordial perturbations, we should not interpret the COBE results to mean that the spectrum must approach something like the Harrison-Zel'dovich form at large scales. Therefore, we also consider a third spectrum, based on a fit to the QDOT data at small scales as well, but which is more sharply cut-off at large scales, with $P(k) \propto k^4$ as $k \rightarrow 0$.

The three spectra we use are

$$P(k) \propto \begin{cases} kT^2(k) & \text{CDM} \\ ke^{-kp} / (1 + (k/k_0)^2) & \text{MGSS-I} \\ k^4 e^{-kp} / (1 + (k/k_0)^5) & \text{MGSS-II} \end{cases} \quad (4.1)$$

Here $T(k)$ is the CDM transfer function of Bond and Efstathiou (1984) for $\Omega = 1$, $\Omega_B = 0.03$, and $h = 0.5$,

$$T(k) = \left[1 + \left(ak + (bk)^{3/2} + (ck)^2 \right)^\nu \right]^{-1/\nu}, \quad (4.2)$$

where

$$a = 5.8(\Omega h^2)^{-1} \text{Mpc}, \quad b = 2.9(\Omega h^2)^{-1} \text{Mpc}, \quad c = 1.6(\Omega h^2)^{-1} \text{Mpc}, \quad \nu = 1.25, \quad (4.3)$$

and $p = 8h^{-1} \text{Mpc}$, $k_0^{-1} = 30h^{-1} \text{Mpc}$. The last two spectra in this list were used by Martinez-Gonzalez, Sanz, and Silk (1992) as approximate phenomenological fits to the power spectrum from the QDOT survey of IRAS galaxies (Kaiser *et al.* 1991). These models, along with the power spectra inferred from the QDOT (Feldman, Kaiser, and Peacock, in preparation) and 1.2 Jansky IRAS redshift catalogs (Fisher *et al.* 1992),

are shown in Fig. 3. We normalize the model spectra in the usual way, by setting the rms mass fluctuation within spheres of radius $8 h^{-1} \text{Mpc}$ to be

$$\sigma_8^2 = \left\langle \left(\frac{\delta M}{M} \right)^2 \right\rangle_{R=8h^{-1}\text{Mpc}} = \frac{1}{2\pi^2} \int_0^\infty dk k^2 P(k) W^2(kR) |_{R=8h^{-1}\text{Mpc}}, \quad (4.4)$$

where the window function is $W(kR) = 3(\sin kR - kR \cos kR)/(kR)^3$. Below, we present results for $\mathcal{T}\sigma_8^{-2}$.

In Fig. 4, we show the anisotropy expected for a 2-beam experiment at a given angular scale, $(\Delta T/T)_\theta^2/2 = (C(0) - C(\theta))$ in units of σ_8^2 , for the linear late-time model f_k^{lin} of eqn.(2.20) and for primordial adiabatic perturbations, for the three spectra of eqn. (4.1). For the MGSS-1 and CDM spectra, which both approach the Harrison-Zel'dovich form at large scales, the large-angle rms anisotropy $(\Delta T/T)_\theta$ in the late-time model is roughly three times larger than the corresponding anisotropy in the model with primordial adiabatic perturbations. The results are different for the sharply falling spectrum MGSS-II: since this spectrum has no power on large scales, the late-time anisotropy is smaller than the primordial Sachs-Wolfe anisotropy in this case, but by less than a factor of 2; this is in agreement with the expectation from eqn. (2.22). Note also that, in accord with Fig. 1, for the same power spectrum $P(k)$, the angular dependence of the anisotropy for the linear late-time model differs substantially from that for primordial fluctuations at small angles.

In Figure 5, we show the correlation function $C(0, \sigma)$ as a function of beam-width σ , with the monopole and dipole terms subtracted off. Again we use the spectra of (4.1), and show results for primordial adiabatic perturbations (eqns. 2.35, 3.11, and 3.12) and for late-time perturbations, eqn. (3.13), minimized according to eqn.(3.7). A note of caution in reading the late-time curves in this Figure: the integral $I_k(0, \sigma)$ has been minimized independently at each value of σ , i.e., different potential functions $f_k(\eta)$ have been chosen at each σ . Therefore, a given late-time curve in this figure does *not* correspond to a fixed late-time scenario (i.e., to a fixed set of $f_k(\eta)$), but rather to many different scenarios. Consequently, the σ -dependence of the late-time curves should not be interpreted as implying that the anisotropy for a given late-time scenario falls off with increasing beam-width according to these curves. In fact, for a *fixed* late-time model that minimizes the signal at some particular σ_c , the fall-off at $\sigma > \sigma_c$ would be more gradual than in the figure, while the rise in the signal at $\sigma < \sigma_c$ would be steeper.

The COBE DMR result for the fluctuation on 10 degrees, $C_{\text{DMR}}(0, \sigma[10^\circ]) = (1.2 \pm 0.4) \times 10^{-10}$, is shown for comparison (Smoot *et al.* 1992). For beam-widths less than a few degrees, the minimal late-time anisotropy is comparable to the primordial adiabatic result. However, at larger beam-widths, the minimal late-time result falls sharply below the Sachs-Wolfe anisotropy. We can understand this result heuristically as follows. For very large beamwidth σ , the contribution of higher multipoles to the sum in (3.13) is strongly suppressed. As a result, in this limit, the anisotropy is dominated by the quadrupole (and perhaps the octopole). However, from the discussion following (2.34), it is clear that one can choose a set of functions $f_k(\eta)$ to make a particular multipole C_ℓ , e.g., the quadrupole, vanish. For η_f sufficiently small, one would normally expect this to produce large values for the other multipoles, but these higher moments enter the large- σ anisotropy with very small weighting. As a consequence, the minimal anisotropy for large beam-widths can be quite small.

V. Conclusion

We have seen that the MBR anisotropy signature of post-recombination structure formation generally differs from that of primordial fluctuations. Consequently, once MBR anisotropy experiments and large-scale structure observations begin to overlap significantly in the lengthscales they probe, comparison of the two would allow one to definitively test whether the fluctuations are primordial or more recent in origin. We have found that, for a given amplitude of present large-scale structure $P(k)$, the minimal late-time anisotropy can be up to an order of magnitude smaller than the corresponding Sachs-Wolfe anisotropy for primordial adiabatic perturbations. However, as comparison of Figs. 4, 5, and 6 show, this minimum is only achieved if we allow sufficiently pathological time dependence for the gravitational potential $\Phi_k(\eta)$. If we restrict the time-dependence of the potential to more well-behaved forms more plausibly to be expected in late-time scenarios (e.g., the linear model or the high-redshift, rapid turn-on model of §II), then the large-angle anisotropy in the late-time scenario is generally comparable to or larger than that due to primordial fluctuations. In particular, this will be the case if the present density fluctuations have substantial power on scales larger than $k^{-1} \simeq 53h^{-1}$ Mpc (Cf. eqn. 2.22). While this result runs counter to part of the motivation for late-time phase transitions, it is not necessarily a negative result for them, given that large-scale MBR anisotropies have now been observed. Furthermore, as recent work suggests (Frieman, Watkins, and Hill, in preparation), perhaps the most likely role for late-time transitions is to amplify perturbations that were initially present over some range of wavelength, rather than to replace primordial fluctuations entirely. In this case, the final power spectrum is due to a combination of primordial and late-time effects, and the induced anisotropy will correspondingly arise from both.

We should also comment on the relation of our work to the recent paper by Martinez-Gonzalez, Sanz, and Silk (1992). These authors calculate the contribution from the time-varying gravitational potential to the anisotropy, as do we. However, they considered a very specific mechanism for the time evolution, namely, that the potential varies in time due to mild nonlinearity of the density inhomogeneities; this effect occurs even in an Einstein-de Sitter (spatially flat) cosmology. The anisotropy induced from this non-linear gravitational evolution is small, $\delta T/T \sim 10^{-6}$, and can be more than an order of magnitude below that arising from primordial adiabatic fluctuations. This small number is not to be compared with ours, since to the second order effect they have calculated must be added either: 1) the effects associated with growing the perturbation to the amplitude at which second order effects become important, or 2) the primordial anisotropy from last scattering. For the standard gravitational instability scenario, effect (2) dominates over the second order effect calculated by Martinez-Gonzalez, et al., unless the universe is reionized after recombination. For late-time scenarios in which (2) is absent or negligible, we have shown in this paper that effect (1) is not necessarily very small and that, in ‘realistic’ models, it is likely to dominate over the second order gravitational contribution to the anisotropy.

Finally, it is worth noting that our methods could be extended or applied in a number of ways. For example, one could use them to estimate the expected anisotropy in a specific late-time scenario, in topological defect models of structure formation, and in the loitering universe model (Sahni, Feldman, and Stebbins 1992, Feldman and Evrard 1992). In addition, one could consider models in which the perturbations induced at late times are non-Gaussian, which one might expect to be a natural outcome of late-time transitions.

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Appendix A

We evaluate here several of the integrals in the text. First we compute the angular average in eqn. (2.8). Consider two unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ separated by angle α , $\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha$, and a wavevector $\hat{\mathbf{k}}$ such that $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \mu = \cos \beta$ and $\hat{\mathbf{m}} \cdot \hat{\mathbf{k}} = \sin \alpha \sin \beta \cos \psi + \cos \alpha \cos \beta$. Then

$$\begin{aligned}
 \langle e^{i\mathbf{k} \cdot (a\hat{\mathbf{n}} - b\hat{\mathbf{m}})} \rangle_{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha} &= \frac{\int d\cos \beta d\psi \exp[i\mathbf{k} \cdot (a\hat{\mathbf{n}} - b\hat{\mathbf{m}})]}{\int d\cos \beta d\psi} \\
 &= \frac{1}{4\pi} \int d\cos \beta d\psi \exp[ik(a \cos \beta - b \sin \alpha \sin \beta \cos \psi - b \cos \alpha \cos \beta)] \\
 &= \frac{1}{4\pi} \int d\cos \beta \exp[ik(a - b \cos \alpha) \cos \beta] \int d\psi \exp(-ikb \sin \alpha \sin \beta \cos \psi) \quad (\text{A.1}) \\
 &= \frac{1}{4\pi} \int d\cos \beta \exp[ik(a - b \cos \alpha) \cos \beta] \times 2\pi J_0(bk \sin \alpha \sin \beta) \\
 &= \int_0^1 d\mu \cos[\mu k(a - b \cos \alpha)] J_0(bk \sin \alpha \sqrt{1 - \mu^2})
 \end{aligned}$$

Doing the remaining integral and applying to the expression (2.8) gives

$$\langle e^{-i\mathbf{k} \cdot \hat{\mathbf{n}}(\eta_0 - \eta) + i\mathbf{k}' \cdot \hat{\mathbf{m}}(\eta_0 - \eta')} \rangle_{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos \alpha} = j_0(k\sqrt{(\eta_0 - \eta)^2 + (\eta_0 - \eta')^2 - 2(\eta_0 - \eta)(\eta_0 - \eta') \cos \alpha}), \quad (\text{A.2})$$

where $j_0(x) = \sin x/x$.

For the integral I_k^{lin} in eqn.(2.23), we have

$$\begin{aligned}
 I_k^{\text{lin}} &= \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x - y))}{k\eta_d(x - y)} \\
 &= \frac{1}{2} \int_{-1}^1 dv \int_{|v|}^{2-|v|} du \frac{\sin(k\eta_d v)}{k\eta_d v} \quad u = x + y, \quad v = x - y \\
 &= \int_{-1}^1 dv (1 - |v|) \frac{\sin(k\eta_d v)}{k\eta_d v} \\
 &= 2 \left[\frac{k\eta_d \text{Si}(k\eta_d) - (1 - \cos(k\eta_d))}{(k\eta_d)^2} \right] \quad (\text{A.3})
 \end{aligned}$$

Next consider the integrals I_1 and I_2 in eqns.(2.27) and following. We have

$$\begin{aligned}
 I_1 &= 2\pi \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x - y))}{k\eta_d(x - y)} (\cos(2\pi x) + \cos(2\pi y)) \\
 &= 2\pi \int_{-1}^1 dv \int_{|v|}^{2-|v|} du \frac{\sin(k\eta_d v)}{k\eta_d v} \cos(\pi u) \cos(\pi v) \quad u = x + y, \quad v = x - y \\
 &= -4 \int_{-1}^1 dv \frac{\sin(k\eta_d v)}{k\eta_d v} \cos(\pi v) \sin(\pi|v|) \\
 &= -4 \int_0^1 dv \frac{\sin(k\eta_d v)}{k\eta_d v} \sin(2\pi v) \\
 &= 2 \left[\frac{\text{Ci}(|k\eta_d + 2\pi|v) - \text{Ci}(|k\eta_d - 2\pi|v)}{k\eta_d} \right]_{v=0}^{v=1} \\
 &= \frac{2}{k\eta_d} \left(\text{Ci}(|k\eta_d + 2\pi|) - \text{Ci}(|k\eta_d - 2\pi|) - \ln \left| \frac{k\eta_d + 2\pi}{k\eta_d - 2\pi} \right| \right) \\
 &\rightarrow -\frac{1}{3\pi} (k\eta_d)^2 + \mathcal{O}((k\eta_d)^4) \quad (\text{A.4})
 \end{aligned}$$

and

$$\begin{aligned}
I_2 &= (2\pi)^2 \int_0^1 dx \int_0^1 dy \frac{\sin(k\eta_d(x-y))}{k\eta_d(x-y)} \cos(2\pi x) \cos(2\pi y) \\
&= 2\pi^2 \int_{-1}^1 dv \int_{|v|}^{2-|v|} du \frac{\sin(k\eta_d v)}{k\eta_d v} \cos(\pi(u+v)) \cos(2\pi(u-v)) \quad u = x+y, \quad v = x-y \\
&= \pi^2 \int_{-1}^1 dv \frac{\sin(k\eta_d v)}{k\eta_d v} \left(-\frac{1}{\pi} \sin(2\pi|v|) + 2(1-|v|) \cos(2\pi v) \right) \\
&= \pi \left[\frac{\text{Ci}(|k\eta_d + 2\pi|v) - \text{Ci}(|k\eta_d - 2\pi|v)}{k\eta_d} \right]_{v=0}^{v=1} + 2\pi^2 \frac{\text{Si}(k\eta_d + 2\pi) + \text{Si}(k\eta_d - 2\pi)}{k\eta_d} \\
&\quad - 4\pi^2 \frac{1 - \cos(k\eta_d)}{(k\eta_d)^2 - (2\pi)^2} \\
&= \frac{\pi}{k\eta_d} \left(\text{Ci}(|k + 2\pi|) - \text{Ci}(|k - 2\pi|) - \ln \left| \frac{k + 2\pi}{k - 2\pi} \right| \right) + \frac{2\pi^2}{k\eta_d} (\text{Si}(k + 2\pi) + \text{Si}(k - 2\pi)) \\
&\quad + \frac{(2\pi)^2}{(2\pi)^2 - (k\eta_d)^2} (1 - \cos(k\eta_d)) \\
&\rightarrow \frac{1}{20\pi^2} (k\eta_d)^4 + \mathcal{O}((k\eta_d)^6)
\end{aligned} \tag{A.5}$$

In the above expressions, we have used the sine and cosine integrals,

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{Ci}(x) = - \int_x^\infty \frac{\cos t}{t} dt, \tag{A.6}$$

Appendix B. Numerical Minimization of the Integral I_k

We wish to find the function $F(x) = f_k(x\eta_t)$ which minimizes the integral

$$I = \int_0^1 dx \int_0^1 dx' G(x, x') \dot{F}(x) \dot{F}(x') \quad (\text{B.1})$$

with the boundary condition that

$$F(0) = 0 \quad F(1) = 1. \quad (\text{B.2})$$

It will be useful to have the symmetry $G(x, x') = G(x', x)$ in this integral. The function $G(,)$ need not be symmetric with respect to its two arguments, but clearly the integral only depends on the symmetric part. If $G(,)$ is not symmetric, then we can use instead

$$\bar{G}(x, x') \rightarrow \frac{1}{2}(G(x, x') + G(x', x)) \quad (\text{B.3})$$

which has the required symmetry. We may integrate (B.1) by parts to obtain

$$\begin{aligned} I &= G(1, 1) - \int_0^1 dx G_{,1}(x, 1) f(x) - \int_0^1 dx' G_{,2}(1, x') f(x') + \int_0^1 dx \int_0^1 dx' G_{,12}(x, x') f(x) f(x') \\ &= \bar{G}(1, 1) - 2 \int_0^1 dx \bar{G}_{,1}(x, 1) f(x) + \int_0^1 dx \int_0^1 dx' \bar{G}_{,12}(x, x') f(x) f(x') \end{aligned} \quad (\text{B.4})$$

where $,1$ denotes differentiation with respect to the first argument and $,2$ the second.

Trapezoidal Rule Discetization

We can discretize this by only specifying $F(x)$ at a finite ordered set of points $\{x_i\}$ for $i = 0, \dots, N+1$ with $x_0 = 0$ and $x_{N+1} = 1$. First consider the trapezoidal rule approximation to (B.1):

$$I = \sum_{i=0}^{N+1} \sum_{j=0}^{N+1} G_{ij} \dot{F}_i \Delta x_i \dot{F}_j \Delta x_j \quad (\text{B.5})$$

where $F_i = F(x_i)$, and

$$G_{ij} = \bar{G}(x_i, x_j) \quad \Delta x_i = \begin{cases} \frac{1}{2}x_1 & i=0 \\ \frac{1}{2}(x_{i+1} - x_{i-1}) & 1 \leq i \leq N \\ \frac{1}{2}(1 - x_N) & i = N+1 \end{cases} \quad \dot{F}_i = \begin{cases} \frac{1}{2\Delta x_1}(F_1 - F_0) & i=0 \\ \frac{1}{2\Delta x_i}(F_{i+1} - F_{i-1}) & 1 \leq i \leq N \\ \frac{1}{2\Delta x_{N+1}}(F_{N+1} - F_N) & i = N+1 \end{cases} \quad (\text{B.6})$$

Then using the notation

$$\Delta F_i = \begin{cases} F_1 - F_0 & i=0 \\ (F_{i+1} - F_{i-1}) & 1 \leq i \leq N \\ (F_{N+1} - F_N) & i = N+1 \end{cases} \quad (\text{B.7})$$

and the fact that $F_0 = 0$ and $F_{N+1} = 1$ to rewrite (B.5) and then “difference by parts” to obtain the “difference” analog of (B.4):

$$I = \frac{1}{4} \sum_{i=0}^{N+1} \sum_{j=0}^{N+1} G_{ij} \Delta F_i \Delta F_j = \sum_{i=1}^N \sum_{j=1}^N M_{ij} F_i F_j - 2 \sum_{i=1}^N B_i F_i + C \quad (\text{B.8})$$

where

$$\begin{aligned} M_{ij} &= \frac{1}{4}(G_{i+1\ j+1} - G_{i+1\ j-1} - G_{i-1\ j+1} + G_{i-1\ j-1}) \\ B_i &= \frac{1}{4}(G_{i+1\ N+1} + G_{i+1\ N} - G_{i-1\ N+1} - G_{i-1\ N}) \\ C &= \frac{1}{4}(G_{NN} + G_{N+1\ N} + G_{N\ N+1} + G_{N+1\ N+1}) \end{aligned} \quad (\text{B.9})$$

One may then use standard linear algebra techniques to solve for the set of $\{F_i\}$ which extremizes this discretized approximation to I :

$$F_i^{\min} = \sum_{j=1}^N M_{ij}^{-1} B_j. \quad (\text{B.10})$$

For large enough N , the set $\{F_i^{\min}\}$ should give a good approximation to the function $F(x)$ which extremizes I . Note that for $\{F_i^{\min}\}$ to actually be a minimum and not a saddle-point requires M_{ij} to have only positive eigenvalues. While it is clear that this is true of the continuous integral operator, we have not shown it for this discretized representation. Also note that, instead of the trapezoidal rule, we could use some higher order approximation to the integral. As long as this approximation is bilinear in F_i , this would just correspond to a different matrix G_{ij} , and the rest of the analysis would carry through.

With the technique described above, we generally find that the set of $\{F_i^{\min}\}$ is discontinuous at the scale of the grid. The discontinuous nature of the minimal $\{F_i^{\min}\}$ holds even for small n , so we are confident that this behavior is not a result of round-off error. The convergence of I for increasing N is also fairly rapid, although there is significant “noise” in this convergence. We cannot be certain that the minimal I we obtain with this method is obtainable with any smooth function $F(x)$. While there is a piecewise bilinear 2-dimensional integrand which gives this integral, we cannot be sure it is of the form $G_{12}(x, x') F(x) F(x')$. We also cannot be certain that M_{ij} has no negative eigenvalues and that the “extremum” we have found is not actually a saddle point. However, we believe that the the minimal values we obtain are actual lower limits to what is achievable with a smooth function. In any case, most of the peculiarities of these results are almost certainly dependent on the difference scheme used. The basic problem is that the integration scheme requires the function to be smooth on the grid scale in order to be accurate, while the minimization scheme is forcing the integrand toward discontinuity. The solution to this problem is to have two resolution scales, the smaller one used to perform the integral and the larger one setting a bound on the jaggedness of $F(x)$. This will allow us to vary the accuracy of the integral and the smoothness of $F(x)$ independently.

Sine Wave Expansion

One way to implement a smoothness cutoff, independently of the size of the grid, is to first expand $F(x)$ in some set of smooth, linearly independent functions. The simplest example are sine functions,

$$F(x) = x + \sum_{a=1}^n \alpha_a \sin a\pi x \quad (\text{B.11})$$

for integer a . Since $\sin a\pi x = 0$ for $x = 0, 1$, eq. (B.11) enforces the boundary condition (B.2). The $\sin a\pi x$ for integral a are a complete set of linearly independent functions on the interval $(0, 1)$, and will be linearly independent of the function x for finite n . However, since x and the $\{\sin a\pi x\}$ are nearly not linearly independent for large n (i.e., smoothed over an interval of size $\sim 1/n$, one can approximate x by a superposition of n sine waves), one must be careful about unnecessary small eigenvalues in the matrix L_{ab} below, which may cause problems numerically. An expansion in Legendre polynomials such as

$$F(x) = P_1(x) + \sum_{a=1}^n \alpha_a (P_{2a+1}(x) - P_1(x)) \quad (\text{B.11}')$$

would not have this problem. However, sines are superior to Legendre polynomials, because they have a fairly uniform variation over the interval and hence give a fairly uniform resolution. The P_l 's for large l have much more rapid variation near $x = 1$ than near $x = 0$.

Using a grid, $\{x_i\}$, as above and the sine wave expansion (B.11), we may rewrite (B.9) as

$$I = \sum_{a=1}^n \sum_{b=1}^n L_{ab} \alpha_a \alpha_b - 2 \sum_{a=1}^n \beta_a \alpha_a + D \quad (\text{B.12})$$

where

$$\begin{aligned} L_{ab} &= \sum_{i=1}^N \sum_{j=1}^N M_{ij} s_a^i s_b^j & \beta_a &= \sum_{i=1}^N B_i s_a^i - \sum_{i=1}^N \sum_{j=1}^N M_{ij} x_i s_a^j \\ D &= \sum_{i=1}^N \sum_{j=1}^N M_{ij} x_i x_j - 2 \sum_{i=1}^N B_i x_i + C & s_a^i &= \sin a\pi x_i \end{aligned} \quad (\text{B.13})$$

This has extremum

$$\alpha_a^{\min} = \sum_{b=1}^n L_{ab}^{-1} \beta_b \quad F^{\min}(x) = x + \sum_{a=1}^n \alpha_a^{\min} \sin a\pi x. \quad (\text{B.14})$$

Of course, we must have $n \leq N$ for L_{ab} to be non-singular. If $n = N$, then L_{ab} is related to M_{ab} by a similarity transformation, and nothing has been changed. However, for $n < N$ we prevent $F(x)$ from varying wildly on the grid scale. In Fig. 6, we show the function $f_k(\eta)$ for the solution which minimizes the integral $I_k(0, \sigma[10^\circ])$ (Cf. 3.13). Here, we have fixed $k\eta_0 = 20$ and $N = 200$ grid points, and we vary the number of sine waves used, $n = 2, 10, 99$, and 100 . We see that, as n is increased, the function f_k becomes increasingly noisy, and that, even at large n , the form of the potential function can change substantially with a small increment in the number of sine waves. Nevertheless, for sufficiently large n , the value of the integral I_k converges fairly well. Comparison of the integral for large and small n shows that the linear model $n = 0$ overestimates I_k by more than an order of magnitude.

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Figure Captions

Fig. 1: The functions $J(k, \alpha)$ (eqn. 2.25) as a function of $k\eta_0$ for the primordial Sachs-Wolfe (SW) anisotropy (dotted curves) and for the linear late-time model (LT), f_k^{lin} (solid curves). Moving from bottom to top, the curves correspond to angles $\alpha = 10, 20, 90$, and 180 degrees.

Fig. 2: The integrals I_k for $C(0, \sigma[1^\circ])$ and $C(0, \sigma[10^\circ])$ for primordial adiabatic perturbations (SW curves) and for the late-time scenarios (LT points) which minimize the correlation function filtered through these two beam-widths, cf. eqn.(3.13). Note that the monopole and dipole terms have been removed.

Fig. 3: Density power spectra $P(k)$ (eqn. 4.1) are shown for the three models of eqn.(4.1), denoted MGSS-1 (solid curve), MGSS-2 (dot-dash curve), and CDM (dashed curve), all normalized to $\sigma_8 = 1$. Also shown are the inferred galaxy power spectra from the QDOT (open squares, Kaiser, et al. 1991, as reanalyzed by Feldman and Kaiser, in preparation) and 1.2 Jansky (crosses, from Fisher, et al. 1992) redshift surveys based on the IRAS catalog. The survey spectra have *not* been corrected for redshift distortions, and are shown here principally to motivate the phenomenological fits of eqn.(4.1). (Note that our convention results in a factor $(\pi/2)^3$ difference in the value of $P(k)$ from that used by Fisher, et al. (1992) for the 1.2 Jansky results.)

Fig. 4: The temperature correlation function $[C(0) - C(\theta)]\sigma_8^{-2}$ vs. θ is shown for the linear late-time model f_k^{lin} of eqn.(2.20) (curves marked by crosses, triangles, and boxes) and for primordial adiabatic fluctuations (unadorned curves), for the three phenomenological spectra of eqn.(4.1).

Fig. 5: The temperature correlation function $C(0, \sigma[\text{FWHM}])\sigma_8^{-2}$ for an experiment of Gaussian beamwidth σ is plotted as a function of beam FWHM, for primordial adiabatic perturbations (unadorned curves) and for the late-time scenario (curves marked by crosses, triangles, and boxes) numerically minimized according to eqn. (3.7) (Cf. eqn.3.13), again for the three spectra of (4.1). Note that each late-time curve corresponds to many different late-time models, each of which minimizes the anisotropy at a given σ . The COBE observation at FWHM of 10 degrees is shown by the closed circle.

Fig. 6: The potential function $f_k(\eta)$ as a function of conformal time η/η_0 is shown for the late-time model that minimizes the 10° anisotropy $C(0, \sigma[10^\circ])$, for fixed wavenumber $k\eta_0 = 20$, $N = 200$ grid points η_i , and for $n = 2, 10, 99$, and 100 sine waves. This demonstrates that the time dependence of the gravitational potential becomes increasingly noisy for large n , but that the integral I_k converges as the number of waves is increased. Note that the boundary conditions $f_k(0) = 0$, $f_k(\eta_f = 0.5\eta_0) = 1$ have been imposed.











